

# LIPSCHITZ AND PATH ISOMETRIC EMBEDDINGS OF METRIC SPACES

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**ABSTRACT.** We prove that each sub-Riemannian manifold can be embedded in some Euclidean space preserving the length of all the curves in the manifold. The result is an extension of Nash  $C^1$  Embedding Theorem. For more general metric spaces the same result is false, e.g., for Finsler non-Riemannian manifolds. However, we also show that any metric space of finite Hausdorff dimension can be embedded in some Euclidean space via a Lipschitz map.

A map  $f : X \rightarrow Y$  between two metric spaces  $X$  and  $Y$  is called a *path isometry* (probably a better name is a *length preserving map*) if, for all curves  $\gamma$  in  $X$ , one has

$$L_Y(f \circ \gamma) = L_X(\gamma).$$

Here  $L_X$  and  $L_Y$  denote the lengths of the parametrized curves with respect to the distances of  $X$  and of  $Y$ , respectively. From the definition, a path isometry is not necessarily injective.

The first aim of the following paper is to show that any sub-Riemannian manifold can be mapped into some Euclidean space via a path isometric embedding, i.e., a topological embedding that is also a path isometry. Sub-Riemannian manifolds are metric spaces when endowed with the Carnot-Carathéodory distance  $d_{CC}$  associated to the fixed sub-bundle and Riemannian structure. For an introduction to sub-Riemannian geometry see [Bel96, Gro99, BBI01, Mon02, Bul02, LD10].

An equivalent statement of our first result is the following. Denote by  $\mathbb{E}^k$  the  $k$ -dimensional Euclidean space. Our result says that, for every sub-Riemannian manifold  $(M, d_{CC})$ , there exists a path connected subset  $\Sigma \subset \mathbb{E}^k$ , for some  $k \in \mathbb{N}$ , such that, when  $\Sigma$  is endowed with the path distance  $d_\Sigma$  induced by the Euclidean length, then the metric space  $(\Sigma, d_\Sigma)$  is isometric to  $(M, d_{CC})$ .

After such a fact one should wonder which are the length metric spaces obtained as subsets of  $\mathbb{E}^k$  with induced length structure. We show that any distance on  $\mathbb{R}^n$  that is coming from a norm but not from a scalar product cannot be obtained in such a way.

We conclude the paper by showing another positive result for general metric spaces: every metric space of finite Hausdorff dimension has a Lipschitz embedding into some  $\mathbb{E}^k$ .

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## 1. INTRODUCTION

In 1954 John Nash showed that any Riemannian manifold can be seen as a  $C^1$  sub-manifold of the Euclidean space. Namely, for any  $n$ -dimensional Riemannian manifold  $(M, g)$ , there exists a  $C^1$  sub-manifold  $N$  of the  $(2n + 1)$ -dimensional Euclidean space  $\mathbb{E}^{2n+1}$  such that  $N$ , endowed with the restriction of the Euclidean Riemannian tensor, is  $C^1$  equivalent to  $(M, g)$ . Two Riemannian manifolds  $(M_1, g_1)$  and  $(M_2, g_2)$  are considered  $C^1$  equivalent if there exists a  $C^1$  diffeomorphism  $f : M_1 \rightarrow M_2$  such that the pulled back tensor  $f^*g_2$  equals  $g_1$ . In Riemannian geometry, a  $C^1$  map  $f$  between two Riemannian manifolds  $(M_1, g_1)$  and  $(M_2, g_2)$  with the property that

$$f : (M_1, g_1) \rightarrow (f(M_1), g_2|_{T(f(M_1))})$$

is a  $C^1$  equivalence is said ‘isometric embedding’. However, in the present paper we will avoid such a term for the reason that the notion of isometric embedding is different in the setting of metric spaces. Indeed, let  $d_{g_1}$  and  $d_{g_2}$  be the distance functions on  $M_1$  and  $M_2$ , respectively, induced by  $g_1$  and  $g_2$ , respectively. Then the fact that  $f : (M_1, g_1) \rightarrow (M_2, g_2)$  is a ‘Riemannian’ isometric embedding does not imply that  $f : (M_1, d_{g_1}) \rightarrow (M_2, d_{g_2})$  is an isometric embedding of the metric space  $(M_1, d_{g_1})$  into the metric space  $(M_2, d_{g_2})$ , i.e., it is not true in general that

$$d_{g_2}(f(p), f(q)) = d_{g_1}(p, q), \quad \forall p, q \in M_1.$$

However, an elementary but important consequence of having a Riemannian isometric embedding is that the length of paths is preserved. In other words, Nash Theorem can be restated saying that any Riemannian manifold can be path isometrically embedded into some Euclidean space.

**Definition 1.1** (Path isometric embedding). A map  $f : X \rightarrow Y$  between two metric spaces  $X$  and  $Y$  is called a *path isometric embedding* if it is a topological embedding, i.e., a homeomorphism onto its image, and, for all curves  $\gamma \subset X$ , one has

$$L_Y(f \circ \gamma) = L_X(\gamma).$$

We want to clarify that the above condition is required also for curves of infinite length.

One of the versions of Nash Theorem can be stated as follows.

**Theorem 1.2** (Nash). *Let  $(M, g)$  be a  $C^\infty$  Riemannian manifold of dimension  $n$ . Then there exists a  $C^1$  path isometric embedding*

$$f : (M, d_g) \rightarrow \mathbb{E}^k,$$

with  $k = 2n + 1$ .

The theorem originally appeared in [Nas54], later it was generalized by Nicolaas Kuiper in [Kui55]. Nowadays, Nash-Kuiper  $C^1$  Theorem is known in the following form.

**Theorem 1.3** (Nash-Kuiper  $C^1$  Embedding Theorem). *Let  $(M, g)$  be a  $C^\infty$  Riemannian manifold of dimension  $n$ . If there is a  $C^\infty$  1-Lipschitz embedding*

$$f : (M, d_g) \rightarrow \mathbb{E}^k$$

*into an Euclidean space  $\mathbb{E}^k$  with  $k \geq n + 1$ , then, for all  $\epsilon > 0$ , there exists a  $C^1$  path isometric embedding*

$$\bar{f} : (M, d_g) \rightarrow \mathbb{E}^k,$$

*that is  $\epsilon$ -close to  $f$ , i.e., for any  $p \in M$ ,*

$$d_{\mathbb{E}}(f(p), \bar{f}(p)) \leq \epsilon.$$

In particular, as follows from a result of Nash which extends the Whitney Embedding Theorem, any  $n$ -dimensional Riemannian manifold admits a path isometric  $C^1$  embedding into an arbitrarily small neighborhood in  $(2n + 1)$ -dimensional Euclidean space.

The Nash-Kuiper Theorem has many counter-intuitive implications. For example, it follows that there exist  $C^1$  path isometric embeddings of the hyperbolic plane in  $\mathbb{E}^3$ . Additionally, any closed oriented Riemannian surface can be  $C^1$  path isometrically embedded into an arbitrarily small ball in Euclidean 3-space. Whereas, for curvature reasons, there is no such a  $C^2$ -embedding.

In [Gro86, 2.4.11] Gromov proved that any Riemannian manifold of dimension  $n$  admits a path isometry into  $\mathbb{E}^n$  (notice the same dimension). In a recent paper [Pet10] Petrunin extended Gromov's result to sub-Riemannian manifolds for a more rigid class of maps: the intrinsic isometries. In fact, Petrunin has the merit of pointing out the importance of the fact that any sub-Riemannian distance is a *monotone* limit of Riemannian distances. This observation will be essential in considering limits of Nash's embeddings as we will do in this paper.

For topological reasons, both Gromov's and Petrunin's maps are in general not injective. Our aim is to have path isometries that are also embeddings. Moreover, any path isometric embedding is an intrinsic isometry, cf. Section 3.2. We extend Nash's result to the metric spaces obtained as limit of an increasing sequence of Riemannian metrics on a fixed manifold, e.g., sub-Riemannian manifolds.

**Theorem 1.4** (Path Isometric Embedding). *Let  $M$  be a  $C^\infty$  manifold of dimension  $n$ . Let  $g_m$  be a sequence of Riemannian structures on  $M$  and let  $d_{g_m}$  be the distance function induced by  $g_m$ . Assume that, for all  $p$  and  $q \in M$ ,*

$$d_{g_m}(p, q) \leq d_{g_{m+1}}(p, q).$$

*Assume also that, for all  $p$  and  $q \in M$ , the limit*

$$d(p, q) := \lim_{m \rightarrow \infty} d_{g_m}(p, q)$$

is finite and that the function  $d$  gives a distance that induces the manifold topology on  $M$ . Then there exists a path isometric embedding

$$f : (M, d) \rightarrow \mathbb{E}^k,$$

with  $k = 2n + 1$ .

In Section 3.1 we will recall the general definition of a sub-Riemannian manifold and show that the sub-Riemannian distance function is a pointwise limit of Riemannian distance functions. Then the following fact will be an immediate consequence of the above theorem.

**Corollary 1.5.** *Each sub-Riemannian manifold of topological dimension  $n$  can be path isometrically embedded into  $\mathbb{E}^{2n+1}$ .*

Actually, the proof of Theorem 1.4 gives a more precise result for the dimension of the target.

**Corollary 1.6.** *As in Theorem 1.4, let  $(M, d)$  be a metric space obtained as a limit of an increasing sequence of Riemannian metrics on a manifold of topological dimension  $n$ . Let  $d_{\text{Riem}}$  be some Riemannian distance such that*

$$d_{\text{Riem}} \leq d.$$

*If there exists a  $C^\infty$  1-Lipschitz embedding*

$$f : (M, d_{\text{Riem}}) \rightarrow \mathbb{E}^k$$

*into an Euclidean space  $\mathbb{E}^k$  with  $k \geq n+1$ , then there exists a path isometric embedding*

$$\bar{f} : (M, d_g) \rightarrow \mathbb{E}^k.$$

Consequently, the Heisenberg group endowed with the usual Carnot-Carathéodory metric is isometric to a subset of  $\mathbb{R}^4$  endowed with the path metric induced by the Euclidean distance, cf. Corollary 3.2. Similarly, the Grushin plane can be realized as a subset of  $\mathbb{R}^3$  with the induced path distance.

Our result is not in contrast with the non-biLipschitz-embeddability of Carnot-Carathéodory spaces. Let us recall that it was observed by Semmes, [Sem96, Theorem 7.1], that Pansu's version of Rademacher Differentiation Theorem [Pan89, MM95] implies that a Lipschitz embedding of a sub-Riemannian manifold  $M$  into an Euclidean space cannot be bi-Lipschitz, unless  $M$  is in fact Riemannian. Indeed, in the case of the Heisenberg group  $\mathbb{H}$ , any Lipschitz map collapses in the direction of the center, i.e.,

$$(1.7) \quad \lim_{g \rightarrow e} \frac{\|f(gx) - f(x)\|_{\mathbb{E}}}{d_{CC}(gx, x)} = 0, \quad g \in \text{Center}(\mathbb{H}).$$

From this fact we understand that any path isometric embedding  $f : \mathbb{H} \rightarrow \mathbb{E}^k$ , which is always a Lipschitz map, has the property that, for  $x \in \mathbb{H}$ , as  $g$  goes to the identity

element inside  $\text{Center}(\mathbb{H})$ ,  $f(gx)$  converges to  $f(x)$  in  $\mathbb{E}^k$  faster than  $d_{CC}(gx, x)$ . This last fact does not contradict the existence of curves inside  $f(\mathbb{H})$  from  $f(gx)$  to  $f(x)$  of length exactly  $d_{CC}(gx, x)$ , as the path isometric embedding property would imply.

Also Corollary 1.5 does not give any dimensional contradiction. Indeed, the path metric  $d_\Sigma$  on a subset  $\Sigma \subset \mathbb{E}^k$  is larger than the restriction on  $\Sigma$  of the Euclidean distance. Thus the metric space  $(\Sigma, d_\Sigma)$  can a priori have Hausdorff dimension strictly greater than  $k = \dim_H(\mathbb{E}^k)$ . The embeddings of Corollary 1.5 give non-constructive examples of sets  $\Sigma \subset \mathbb{R}^k$  with the property that

$$\dim_H(\Sigma, d_\Sigma) > k.$$

Notice that for such examples, the metric  $d_\Sigma$  induces on  $\Sigma$  the subspace topology of  $\mathbb{R}^k$ .

For the sake of completeness let us mention the following different generalization by D'Ambra of Nash's result to the case of contact manifolds. Namely, let  $(M_1, \xi_1, g_1)$  and  $(M_2, \xi_2, g_2)$  be two contact manifolds with contact structures  $\xi_1$  and  $\xi_2$ , respectively, and Riemannian metrics  $g_1$  and  $g_2$ , respectively. The main result in [D'A95] claims that if  $\dim(M_2) \geq 2 \dim(M_1) + 3$  and  $M_1$  is compact, then there exists a  $C^1$  embedding

$$f : M_1 \rightarrow M_2,$$

preserving the contact structures and the Riemannian tensors on  $\xi_1$ , i.e.,

$$f_*\xi_1 \subset \xi_2 \quad \text{and} \quad g_1|_{\xi_1} = f^*(g_2|_{f_*\xi_1}).$$

We consider now possible generalizations of Theorem 1.4. We shall observe that it is not true that any finite dimensional metric space admits a path isometric embedding into some Euclidean space. Indeed, there is no path isometry from  $(\mathbb{R}^2, \|\cdot\|_\infty)$  to any  $\mathbb{E}^k$ . Here  $\|\cdot\|_\infty$  is the supremum norm on  $\mathbb{R}^2$ , which is not coming from a scalar product. In general we have the following:

**Proposition 1.8.** *Let  $(M, \|\cdot\|)$  be a Finsler manifold. If there exists a path isometry*

$$f : (M, \|\cdot\|) \rightarrow \mathbb{E}^k,$$

*then the manifold is in fact Riemannian.*

The proof of the above proposition is a consequence of Rademacher Theorem and has been noticed by other authors as well, cf. [Pet10].

An important topological theorem, due to K. Menger and G. Nöbeling, states that any compact metrizable space of topological dimension  $m$  can be embedded in  $\mathbb{R}^k$  for  $k = 2m + 1$ . For a reference, see [Mun75]. We shall show the analogue for Lipschitz

embedding of metric spaces, whose proof is an application of the Baire Category Theorem as well as for the topological version.

**Theorem 1.9** (Lipschitz Embedding). *Any compact metric space of Hausdorff dimension  $m$  can be embedded in  $\mathbb{E}^k$  via a Lipschitz map, for  $k = 2m + 1$ .*

Since compact sub-Finsler manifolds are biLipschitz equivalent to sub-Riemannian manifolds, any sub-Finsler manifold is locally biLipschitz equivalent to a subset of some  $\mathbb{E}^k$  with the path distance. In other words, any sub-Finsler manifold can be embedded into  $\mathbb{E}^k$  via a map that distorts lengths by a controlled ratio. Namely, we already know that for sub-Finsler manifolds the following conjecture holds.

**Conjecture 1.10** (BLD embeddings). *Any compact length metric space of finite Hausdorff dimension can be embedded in some Euclidean space via a bounded-length-distortion map.*

**Definition 1.11** (BLD). A map  $f : X \rightarrow Y$  between two metric spaces  $X$  and  $Y$  is said of *bounded-length-distortion* (BLD for short), if there exists a constant  $C$  such that, for all curves  $\gamma \subset X$ , one has

$$(1.12) \quad C^{-1}L_X(\gamma) \leq L_Y(f \circ \gamma) \leq CL_X(\gamma).$$

We expect the above conjecture to hold, more because of lack of counterexamples than for actual reasoning. The map given by Theorem 1.9 satisfies the upper bound of equation (1.12). However, even if such a map is injective, it might not satisfy the lower bound of equation (1.12).

## 2. EXISTENCE OF PATH ISOMETRIC EMBEDDINGS

**2.1. Preliminaries.** The following Theorem 2.1 might seem an easy corollary of Nash-Kuiper Theorem 1.3. Indeed, by Nash-Kuiper, any smooth 1-Lipschitz embedding is arbitrarily close to a  $C^1$  length-preserving embedding. By smoothing one understands that the following result holds: any smooth 1-Lipschitz embedding is arbitrarily close to a  $C^\infty$  almost-length-preserving embedding. However, the claim of Theorem 2.1 is one of the strategic steps of Nash-Kuiper proof.

**Theorem 2.1** (Consequence of Nash's proof). *Let  $(M, g)$  be a  $C^\infty$  Riemannian manifold. If there is a  $C^\infty$  1-Lipschitz embedding*

$$f : (M, d_g) \rightarrow \mathbb{E}^k$$

*into an Euclidean space  $\mathbb{E}^k$  with  $k \geq n+1$ , then, for any  $a > 0$  and for any continuous function  $b : M \rightarrow \mathbb{R}_{>0}$ , there exists a  $C^\infty$  1-Lipschitz embedding*

$$\bar{f} : (M, d_g) \rightarrow \mathbb{E}^k,$$

*such that, for any curve  $\gamma \subset M$ ,*

$$(1 - a)L_g(\gamma) \leq L_{\mathbb{E}}(\bar{f} \circ \gamma) \leq L_g(\gamma)$$

and, for any  $p \in M$ ,

$$d_{\mathbb{E}}(f(p), \bar{f}(p)) \leq b(p).$$

For compact manifolds the following result is an easy consequence of Whitney Embedding Theorem, where in fact one can take  $k = 2n$ . For general manifolds a proof can be found in [Nas54, page 394].

**Theorem 2.2** (Whitney-Nash). *Let  $(M, g)$  be a  $C^\infty$  Riemannian manifold of dimension  $n$ . Then there exists a  $C^\infty$  1-Lipschitz embedding*

$$f : (M, d_g) \rightarrow \mathbb{E}^k,$$

with  $k = 2n + 1$ .

The following fact is the key for preventing loss of length in the limit process while proving Theorem 1.4. A similar argument was used in [Pet10].

**Definition 2.3** ( $I(\delta)$ ). Let  $f : M \rightarrow \mathbb{R}^k$  be a  $C^\infty$  embedding. Let  $\delta : M \rightarrow \mathbb{R}_{>0}$  be a continuous function. We consider the  $\delta$ -neighborhood of  $f(M)$  as the set

$$I(\delta) := I_\delta(f(M)) := \{x \in \mathbb{R}^k : \|x - f(p)\|_{\mathbb{E}} < \delta(p), \text{ for some } p\}.$$

**Lemma 2.4** (Control on tubular neighborhoods). *Let  $(M, g)$  be a  $C^\infty$  Riemannian manifold. Let*

$$f : M \rightarrow \mathbb{R}^k$$

*be a  $C^\infty$  embedding. Then, for any  $\eta > 0$ , there exists a positive continuous function  $\delta = \delta_{f,\eta} : M \rightarrow (0, \eta)$  such that, for all  $x, y \in M$ , we have that, for all  $x, y \in f(M)$ ,*

$$(1 - \eta)d_{f(M)}(x, y) \leq d_{I(\delta)}(x, y) \leq d_{f(M)}(x, y),$$

*where  $d_{f(M)}$  and  $d_{I(\delta)}$  are the path metrics in  $f(M)$  and  $I(\delta)$ , respectively.*

*Proof of Lemma 2.4.* Roughly speaking, the idea is that, since  $f(M)$  is a  $C^\infty$  submanifold of  $\mathbb{R}^k$ , then the ‘normal projection’ of a small tubular neighborhood of  $f(M)$  in  $\mathbb{R}^k$  onto  $f(M)$  is  $(1 + \epsilon)$ -Lipschitz. Therefore the loss of length is small.

More rigorously, the Neighborhood Theorem, cf. [GP74, page 69], states that there exists a  $C^0$  function  $\delta = \delta_{f,\eta} : M \rightarrow \mathbb{R}_{>0}$  and a submersion

$$\pi : I_\delta(f(M)) \longrightarrow f(M)$$

that is the identity on  $f(M)$ . We can modify  $\delta$  to have the further properties that  $\delta(p) < \eta$ , for all  $p \in M$ , and that  $\pi$  is a  $(1 - \eta)^{-1}$ -Lipschitz map (consequence of the fact that  $\pi$  is 1-Lipschitz on  $f(M)$ ).

Now, obviously  $d_{I(\delta)} \leq d_{f(M)}$ , since  $f(M) \subset I(\delta)$ . For the other inequality, let  $\gamma$  be a curve in  $I(\delta)$  from  $x$  to  $y$ , with  $x, y \in f(M)$ , that is a geodesic for  $d_{I(\delta)}$ . Project

$\gamma$  onto  $f(M)$  via  $\pi$  and get a curve in  $f(M)$  from  $x$  to  $y$ . We therefore estimate the distances as

$$\begin{aligned} d_{I(\delta)}(x, y) &= L_{\mathbb{E}}(\gamma) \\ &\geq (1 - \eta)L_{\mathbb{E}}(\pi \circ \gamma) \\ &\geq (1 - \eta)d_{f(M)}(x, y), \end{aligned}$$

for each  $x, y \in f(M)$ .  $\square$

**2.2. Proof of the existence of path isometric embeddings.** This section is devoted to the proof of Theorem 1.4. We will first construct the map  $f$ , then prove that it is a path isometry, and finally that it is an embedding.

*The construction of  $f$ .* From Theorem 2.2, we can start with a  $C^\infty$  1-Lipschitz embedding

$$f_1 : (M, g_1) \rightarrow \mathbb{E}^k.$$

Set, for  $m \in \mathbb{N}$ , the auxiliary functions:

$$a_m := \frac{1}{m}, \quad \eta_m := \frac{1}{m}.$$

Considering the function  $\delta_{f, \eta}$  of Lemma 2.4, set  $\delta_1 := \delta_{f_1, \eta_1}$ . Choose any  $C^0$  function  $b_1$  with  $0 < b(p) < \delta_1(p)$ , for all  $p \in M$ .

By recurrence, for each  $m \in \mathbb{N}$ , perform the following construction of  $C^\infty$  1-Lipschitz embeddings

$$f_m : (M, g_m) \rightarrow \mathbb{E}^k$$

and positive continuous function  $b_m$  and  $\delta_m$  both smaller than  $1/m$ , such that the following four properties hold:

$$(2.5) \quad \delta_m = \delta_{f_m, \eta_m}, \quad \forall m > 1,$$

$$(2.6) \quad \sum_{i=m}^{\infty} b_i(p) < \delta_m(p), \quad \forall m > 1, \forall p \in M$$

$$(2.7) \quad (1 - a_{m-1})L_{g_m}(\gamma) \leq L_{\mathbb{E}}(f_m \circ \gamma) \leq L_{g_m}(\gamma), \quad \forall \text{ curve } \gamma \subset M, \forall m > 1,$$

$$(2.8) \quad d_{\mathbb{E}}(f_{m-1}(p), f_m(p)) \leq b_{m-1}(p), \quad \forall p \in M, \forall m > 1.$$

Indeed, we already constructed  $f_1$ ,  $b_1$ , and  $\delta_1$ . Assume that, for fixed  $m$ ,  $f_m$ ,  $b_m$ , and  $\delta_m$  have been constructed. Let us construct  $f_{m+1}$ ,  $b_{m+1}$ , and  $\delta_{m+1}$ . Note that, since  $d_{g_m} \leq d_{g_{m+1}}$  and  $f_m : (M, g_m) \rightarrow \mathbb{E}^k$  is 1-Lipschitz, we have that  $f_m : (M, g_{m+1}) \rightarrow \mathbb{E}^k$  is 1-Lipschitz as well. Applying Theorem 2.1 for  $f_m$ ,  $a_m$ , and  $b_m$ , we get a  $C^\infty$  1-Lipschitz embedding  $f_{m+1} : (M, g_{m+1}) \rightarrow \mathbb{E}^k$  such that

$$(1 - a_m)L_{g_{m+1}}(\gamma) \leq L_{\mathbb{E}}(f_{m+1} \circ \gamma) \leq L_{g_{m+1}}(\gamma), \quad \forall \text{ curve } \gamma \subset M,$$



and

$$d_{\mathbb{E}}(f_m(p), f_{m+1}(p)) \leq b_m(p), \quad \forall p \in M.$$

Define  $\delta_{m+1} = \delta_{f_{m+1}, \eta_{m+1}}$ . Finally, notice that the inequalities (2.6) are strict, so we already have that

$$\sum_{i=l}^m b_i < \delta_l, \quad \forall l, j \text{ such that } 1 \leq l \leq j.$$

Therefore we can choose a continuous function  $b_{m+1} : M \rightarrow \mathbb{R}$  with  $0 < b_{m+1} < \delta_{m+1}$  and such that

$$\sum_{i=l}^{m+1} b_i < \delta_l, \quad \forall l, j \text{ such that } 1 \leq l \leq j.$$

The construction of  $\{f_m\}$ ,  $\{b_m\}$ , and  $\{\delta_m\}$  is concluded.

We should notice that from (2.8) and (2.6) we have that, if  $m < j$ ,

$$(2.9) \quad d_{\mathbb{E}}(f_m(p), f_{j+1}(p)) \leq \sum_{i=m}^j b_i(p) \leq \delta_m(p) \leq \eta_m = \frac{1}{m}.$$

In other words, for  $j$  big enough,

$$(2.10) \quad f_j(M) \subset I_{\delta_m}(f_m(M)) \subset \mathbb{E}^k.$$

After having constructed the sequence of approximating maps  $f_m$ , let us consider their limit. Notice that, since  $d_{g_m} \leq d$ , then the maps

$$f_m : (M, d) \rightarrow \mathbb{E}^k$$

are 1-Lipschitz. By (2.9), the maps  $f_m$  converge uniformly to a map

$$f : (M, d) \rightarrow \mathbb{E}^k,$$

which is obviously 1-Lipschitz as well.

*The map  $f$  is a path isometry.* We will prove, that

$$(2.11) \quad L_d(\gamma) \geq L_{\mathbb{E}}(f \circ \gamma), \quad \forall \text{ curve } \gamma \subset M,$$

and that

$$(2.12) \quad L_d(\gamma) \leq L_{\mathbb{E}}(f \circ \gamma), \quad \forall \text{ curve } \gamma \subset M.$$

The fact that (2.11) holds is obvious since  $f$  is 1-Lipschitz with respect to  $d$ . For the proof of (2.12) we have to make use of the fact that  $\delta_m$  have been constructed via the function  $\delta$  of Lemma 2.4. Observe that, taking limit in (2.10), as  $j \rightarrow \infty$ , we have that, for all  $m \in \mathbb{N}$ ,

$$(2.13) \quad f(M) \subset I_{\delta_m}(f_m(M)) \subset \mathbb{E}^k.$$

Let  $I_m := I_{\delta_m}(f_m(M))$ , and let  $d_{I_m}$  be the path metric on it.

In order to prove (2.12), take any curve  $\gamma \subset M$  and take  $p_0, p_1, \dots, p_N \in \gamma$  consecutive points on the curve. Fix one of the indices  $l \in \{1, \dots, N\}$ . Consider the curve

$$\sigma_l := [f_m(p_{l-1}), f(p_{l-1})] \cup f(\gamma|_{[p_{l-1}, p_l]}) \cup [f(p_l), f_m(p_l)],$$

where  $[A, B]$ , with  $A, B \in \mathbb{E}^k$ , is the Euclidean segment connecting  $A$  and  $B$ . By the containment (2.13), we have the containment

$$\sigma_l \subset I_m, \quad \forall m \in \mathbb{N}.$$

In other words, the curve  $\sigma_l$  connects the two points  $f_m(p_{l-1})$  and  $f_m(p_l)$  inside the neighborhood  $I_m$ , so its length is greater than the path distance inside  $I_m$  of such two points, i.e.,

$$d_{I_m}(f_m(p_{l-1}), f_m(p_l)) \leq L_{\mathbb{E}}(\sigma_l).$$

Now, on one hand, by the definition of  $\sigma_l$  we have that

$$L_{\mathbb{E}}(\sigma_l) \leq \delta_m(p_{l-1}) + L_{\mathbb{E}}(f \circ \gamma|_{[p_{l-1}, p_l]}) + \delta_m(p_l) \leq 2\eta_m + L_{\mathbb{E}}(f \circ \gamma|_{[p_{l-1}, p_l]}).$$

On the other hand, Lemma 2.4 says that, since  $\delta_m$  equals  $\delta_{f_m, \eta_m}$ , we have that

$$(1 - \eta_m)d_{f_m(M)}(f_m(p_{l-1}), f_m(p_l)) \leq d_{I_m}(f_m(p_{l-1}), f_m(p_l)).$$

Therefore

$$(1 - \eta_m)d_{f_m(M)}(f_m(p_{l-1}), f_m(p_l)) \leq 2\eta_m + L_{\mathbb{E}}(f \circ \gamma|_{[p_{l-1}, p_l]}).$$

Since  $f_m$  are  $(1 - a_m)$ -almost isometries (in the sense of (2.7)), we get

$$(1 - \eta_m)(1 - a_m)d_{g_m}(p_{l-1}, p_l) \leq 2\eta_m + L_{\mathbb{E}}(f \circ \gamma|_{[p_{l-1}, p_l]}).$$

Summing over  $l$ , we have that

$$(1 - \eta_m)(1 - a_m) \sum_{l=1}^N d_{g_m}(p_{l-1}, p_l) \leq 2\eta_m N + L_{\mathbb{E}}(f \circ \gamma).$$

Now take the limit for  $m \rightarrow \infty$ . Since  $\eta_m \rightarrow 0$ ,  $a_m \rightarrow 0$ , (and note that  $N$  is fixed), we get

$$\sum_{l=1}^N d(p_{l-1}, p_l) \leq L_{\mathbb{E}}(f \circ \gamma).$$

Finally, taking limits on all partitions of points  $\{p_l\}$ , we have that

$$L_d(\gamma) \leq L_{\mathbb{E}}(f \circ \gamma).$$

*The map  $f$  is an embedding.* Assume by contradiction that there exists a point  $q_0 \in N$  and a sequence of points  $q_k \in M$  with

$$f(q_k) \rightarrow f(q_0), \quad \text{but} \quad d(q_0, q_k) > \alpha, \forall k \in \mathbb{N},$$

for some positive value  $\alpha$ . Since  $d$  and  $d_{g_1}$  give the same topology, there exists a  $\beta > 0$  such that

$$B_{d_{g_1}}(q_0, \beta) \subset B_d(q_0, \alpha).$$

Therefore, since the distances  $d_{g_m}$  are increasing, we can take  $m$  large enough such that the following four inequalities hold:

$$(2.14) \quad d_{g_m}(q_0, q_k) > d_{g_1}(q_0, q_k) > \beta, \quad \forall k \in \mathbb{N}$$

$$(2.15) \quad 1 - \eta_m > \frac{1}{2},$$

$$(2.16) \quad \delta_m < \frac{\beta}{16},$$

$$(2.17) \quad 1 - a_m > \frac{1}{2}.$$

Then, on one hand,

$$\begin{aligned} d_{I_m}(f_m(q_k), f_m(q_0)) &\leq d_{I_m}(f(q_k), f(q_0)) + \delta_m(q_k) + \delta_m(q_0) \\ &\leq d_{I_m}(f(q_k), f(q_0)) + 2\eta_m \\ &\leq d_{I_m}(f(q_k), f(q_0)) + \frac{\beta}{8}. \end{aligned}$$

On the other hand,

$$\begin{aligned} d_{I_m}(f_m(q_k), f_m(q_0)) &\geq (1 - \eta_m)d_{f_m(M)}(f_m(q_k), f_m(q_0)) \\ &\geq (1 - \eta_m)(1 - a_m)d_{g_m}(q_k, q_0) \\ &\geq \beta/4. \end{aligned}$$

So we get

$$d_{I_m}(f(q_k), f(q_0)) \geq \frac{\beta}{4} - \frac{\beta}{8} = \frac{\beta}{8} > 0,$$

which contradicts the fact that  $f(q_k) \rightarrow f(q_0)$ , as  $k \rightarrow \infty$ . □

### 3. MORE ON PATH ISOMETRIC EMBEDDINGS

#### 3.1. Sub-Riemannian geometries and the proof of Corollaries 1.5 and 1.6.

**Definition 3.1** (The general definition of sub-Riemannian manifold). A (smooth) sub-Riemannian structure on a manifold  $M$  is a function  $\rho : TM \rightarrow [0, \infty]$  obtained by the following construction: Let  $E$  be a vector bundle over  $M$  endowed with a scalar product  $\langle \cdot, \cdot \rangle$  and let

$$\sigma : E \rightarrow TM$$

be a morphism of vector bundles. For each  $p \in M$  and  $v \in T_p M$ , set

$$\rho_p(v, v') := \inf \{ \langle u, u' \rangle : u, u' \in E_p, \sigma(u) = v, \sigma(u') = v' \}.$$

Define  $\rho_p(v) := \rho_p(v, v)$  and, given an absolutely continuous path  $\gamma : [0, 1] \rightarrow M$ , define

$$L_\rho(\gamma) := \int_0^1 \sqrt{\rho_{\gamma(t)}(\dot{\gamma}(t))} dt.$$

The *sub-Riemannian distance associated to  $\rho$*  is defined as, for any  $p$  and  $q$  in  $M$ ,

$$d_{CC}(p, q) = \inf \left\{ L_\rho(\gamma) \mid \gamma \text{ absolutely continuous path } \gamma(0) = p, \gamma(1) = q \right\}.$$

The only extra assumption on  $\rho$  is that the distance  $d_{CC}$  is finite and induces the manifold topology.

*Proof of Corollary 1.5.* We show now that each sub-Riemannian distance can be obtained as a limit of increasing Riemannian distances. The proof is easy and well-known in the case when  $E$  is in fact a sub-bundle of the tangent bundle. Here we give the proof in the general case.

Let  $\rho : TM \rightarrow [0, \infty]$  be the function defining the sub-Riemannian structure. Notice that  $\rho(v) = 0$  only if  $v = 0$ . So one can take some Riemannian tensor  $g_1$  with the property that  $g_1 \leq \rho$ .

Then, by recurrence, for each  $m \in \mathbb{N}$ , we consider  $g_m$  to be a (smooth) Riemannian tensor with the property that, at any point  $p \in M$ ,

$$\max\{(g_{m-1})_p(v, w), \min\{(1 - 2^{-m})\rho_p(v, w), m(g_1)_p(v, w)\}\} \leq (g_m)_p(v, w) \leq \rho_p(v, w).$$

Obviously we have that

$$g_1 \leq g_m \leq g_{m+1} \leq \rho.$$

Then, for any absolutely continuous path  $\gamma$ , we have that

$$L_{g_m}(\gamma) \leq L_\rho(\gamma).$$

Thus, for any  $p$  and  $q$  in  $M$ ,

$$d_{g_m}(p, q) \leq d_{CC}(p, q),$$

and therefore

$$\lim_{m \rightarrow \infty} d_{g_m}(p, q) \leq d_{CC}(p, q).$$

Assume, by contradiction, that, for some  $p$  and  $q$  in  $M$ , we have that

$$\lim_{m \rightarrow \infty} d_{g_m}(p, q) < d_{CC}(p, q).$$

Then there are curves  $\gamma_m$  from  $p$  to  $q$  such that

$$\lim_{m \rightarrow \infty} L_{g_m}(\gamma_m) < d_{CC}(p, q).$$

Since

$$L_{g_1}(\gamma_m) \leq L_{g_m}(\gamma_m),$$

we get a bound on the lengths  $L_{g_1}(\gamma_m)$ . Therefore, by Ascoli-Arzelà argument,  $\gamma_m$  converges to a curve  $\gamma$  from  $p$  to  $q$ . We may assume that  $\gamma$  is parametrized by arc length with respect to the distance of  $g_1$ . Now, either  $L_\rho(\gamma)$  is infinite or is finite. Namely, either there is a positive-measure set  $A \subset [0, 1]$  such that

$$\rho_{\gamma(t)}(\dot{\gamma}(t)) = \infty, \quad \forall t \in A,$$

or, for almost every  $t \in [0, 1]$ , the value  $\rho_{\gamma(t)}(\dot{\gamma}(t))$  is finite.

In the first case, for all  $t \in A$ ,

$$(g_m)_{\gamma(t)}(\dot{\gamma}(t)) \geq m(g_1)_{\gamma(t)}(\dot{\gamma}(t)).$$

From this we have that

$$L_{g_m}(\gamma) \geq m L_{g_1}(\gamma|_A) \rightarrow \infty, \quad \text{as } m \rightarrow \infty.$$

We get a contradiction since by assumption  $d_{CC}(p, q) < \infty$ .

In the second case, for almost all  $t$ , for  $m$  big enough,

$$(1 - 2^{-m})\rho_{\gamma(t)}(\dot{\gamma}(t)) \leq (g_m)_{\gamma(t)}(\dot{\gamma}(t)) \leq \rho_{\gamma(t)}(\dot{\gamma}(t)).$$

From this we have that

$$L_{g_m}(\gamma) \rightarrow L_\rho(\gamma), \quad \text{as } m \rightarrow \infty.$$

We get a contradiction since we have that  $d_{CC}(p, q) \leq L_\rho(\gamma)$ .  $\square$

*Proof of Corollary 1.6.* Corollary 1.6 is not a direct consequence of the claim of Theorem 1.4. However, the proof is the same. Indeed, in the proof of the theorem we started with the embedding

$$f_1 : (M, g_1) \rightarrow \mathbb{E}^k$$

with  $k = 2n + 1$ , which was given by Theorem 2.2. If instead, as assumed in Corollary 1.6, we already have an embedding

$$f : (M, d_{\text{Riem}}) \rightarrow \mathbb{E}^k$$

with  $k \geq n + 1$ , then we can consider a sequence of increasing Riemannian distances starting with  $d_{g_1} = d_{\text{Riem}}$  and converging pointwise to  $d$ . At each stage, each 1-Lipschitz embedding can be stretched as in Theorem 1.4, since in Theorem 2.1 we only need the codimension to be greater than 1, i.e.,  $k \geq n + 1$ .  $\square$

**Corollary 3.2.** *Let  $(\mathbb{H}, d_{CC})$  be the Heisenberg group endowed with the sub-Riemannian distance with the first layer as horizontal distribution. Then we have that there exists a subset  $\Sigma$  of  $\mathbb{R}^4$ , such that, if  $d_\Sigma$  is the path metric induced by the Euclidean length of  $\mathbb{R}^4$ , then  $(\mathbb{H}, d_{CC})$  is isometric to  $(\Sigma, d_\Sigma)$ .*

*Proof.* The statement is a direct consequence of Corollary 1.6 and Proposition 3.4. We make use of the fact that there is a Lipschitz homeomorphism  $f : (\mathbb{H}, d_{CC}) \rightarrow \mathbb{E}^3$ .  $\square$

*Remark 3.3.* A similar proof gives the following fact. The Grushin plane  $\mathbb{P}$  can be realized as a subset of  $\mathbb{R}^3$  with the induced path distance. The reason is that the identity map from  $\mathbb{P}$  to  $\mathbb{E}^2$  is a Lipschitz embedding. One again concludes using Corollary 1.6 and Proposition 3.4.

### 3.2. Isometries, intrinsic isometries, and path isometries.

**Proposition 3.4.** *Let  $f : (X, d_X) \rightarrow (Y, d_Y)$  be a map between proper geodesic metric spaces. Then  $f$  is a path isometric embedding if and only if the space  $f(X)$  endowed with the path distance  $d_{f(X)}$  induced by  $d_Y$  is isometric to  $(X, d_X)$  via  $f$  and the topology induced by  $d_{f(X)}$  coincided with the topology of  $f(X)$  as a topological subspace of  $Y$ .*

*Proof.* Let us denote by  $\tau_X$  and  $\tau_Y$  the topology of  $(X, d_X)$  and  $(Y, d_Y)$ , respectively. Let  $\tau_{d_{f(X)}}$  be the topology on  $f(X)$  induced by the path distance  $d_{f(X)}$ . We shall write  $A \simeq B$  to say that  $A$  is homeomorphic to  $B$ .

$\Leftarrow$ ] If  $f : (X, d_X) \rightarrow (f(X), d_{f(X)})$  is an isometry, then it preserves the length of paths. Since the length structures on  $f(X)$  and  $Y$  coincide, then  $f : (X, d_X) \rightarrow (Y, d_Y)$  is a path isometry. Moreover, since  $f : (X, d_X) \rightarrow (f(X), d_{f(X)})$  is an isometry, then  $(X, \tau_X) \simeq (f(X), \tau_{d_{f(X)}})$ . If, by assumption  $(f(X), \tau_Y) \simeq (f(X), \tau_{d_{f(X)}})$ , we have that  $(f(X), \tau_Y) \simeq (X, \tau_X)$ , i.e.,  $f$  is an embedding.

$\Rightarrow$ ] If  $f$  is an embedding, we have that  $(f(X), \tau_Y) \simeq (X, \tau_X)$ . Moreover, since  $f$  has a continuous inverse on  $f(X)$ , there is a one-to-one correspondence between curves in  $X$  and curves in  $f(X)$ . If  $f$  is a path isometry, then such a correspondence preserves length. Since both  $d_X$  and  $d_Y$  are length spaces, we have that

$$d_X(x, y) = d_{f(X)}(f(x), f(y)), \quad x, y \in X,$$

i.e.,  $f : (X, d_X) \rightarrow (f(X), d_{f(X)})$  is an isometry.

We also have as a consequence that  $(X, \tau_X) \simeq (f(X), \tau_{d_{f(X)}})$ . If by assumption we have that  $f$  is an embedding, then, by definition, we have that  $(f(X), \tau_Y) \simeq (X, \tau_X)$ . We conclude that  $(f(X), \tau_Y) \simeq (f(X), \tau_{d_{f(X)}})$ .  $\square$

Let  $f : X \rightarrow Y$  be a map between length spaces. Given two points  $p, q \in X$ , a sequence of points  $p = x_0, x_1, \dots, x_N = q$  in  $X$  is called an  $\epsilon$ -chain from  $p$  to  $q$  if  $d(x_{i-1}, x_i) \leq \epsilon$  for all  $i = 1, \dots, N$ . Set

$$\text{pull}_{f, \epsilon}(p, q) = \inf \left\{ \sum_{i=1}^N d(f(x_{i-1}), f(x_i)) \right\}$$

where the infimum is taken along all  $\epsilon$ -chains  $\{x_i\}_{i=0}^N$  from  $p$  to  $q$ . The limit

$$\text{pull}_f(p, q) := \lim_{\epsilon \rightarrow 0} \text{pull}_{f, \epsilon}(p, q)$$

defines a (possibly infinite) pre-metric.

A map  $f : X \rightarrow Y$  is called an *intrinsic isometry* if

$$d(p, q)_X = \text{pull}_f(p, q)$$

for any  $p, q \in X$ .

**Proposition 3.5.** *A path isometric embedding  $f : X \rightarrow Y$  between proper geodesic spaces is an intrinsic isometry.*

*Proof.* Take  $p$  and  $q \in X$ . Let  $\gamma$  be a geodesic from  $p$  to  $q$ . Fix  $\epsilon > 0$ . Let  $t_0 < t_1 < \dots < t_N$  be such that

$$\gamma(t_0) = p, \quad \gamma(t_N) = q,$$

and

$$\{\gamma(t_j)\}_{j=0}^N \text{ is an } \epsilon\text{-chain.}$$

Then, using that  $f$  is a path isometry, we have that

$$\begin{aligned} \text{pull}_{f,\epsilon}(p, q) &\leq \sum_{i=1}^N d(f(\gamma(t_{i-1})), f(\gamma(t_i))) \\ &\leq \sum_{i=1}^N L_Y(f \circ \gamma|_{[t_{i-1}, t_i]}) \\ &= \sum_{i=1}^N L_X(\gamma|_{[t_{i-1}, t_i]}) \\ &= L_X(\gamma) \\ &= d(p, q). \end{aligned}$$

To prove the other inequality, assume by contradiction that there is some  $\alpha > 0$  and there is some  $\epsilon_0 > 0$  such that, for all  $\epsilon \in (0, \epsilon_0)$ , we have that

$$\text{pull}_{f,\epsilon}(p, q) \leq d(p, q) - \alpha.$$

Thus, for each such an  $\epsilon$  there exists an  $\epsilon$ -chain  $\{x_i^{(\epsilon)}\}_{i=0}^N$  from  $p$  to  $q$  with the property that

$$\sum_{i=1}^N d(f(x_{i-1}^{(\epsilon)}), f(x_i^{(\epsilon)})) \leq d(p, q) - \alpha/2.$$

Consider a curve  $\sigma_\epsilon$  in  $Y$  passing through the points  $f(x_0^{(\epsilon)}), f(x_1^{(\epsilon)}), \dots, f(x_N^{(\epsilon)})$  and forming a geodesic between  $f(x_{i-1}^{(\epsilon)})$  and  $f(x_i^{(\epsilon)})$ . Therefore we have that

$$L_Y(\sigma_\epsilon) \leq d(p, q) - \alpha/2.$$

From such a bound on the length, from the fact that  $\sigma_\epsilon$  starts at the fixed point  $f(p)$ , and from the fact that  $Y$  is locally compact, we have that there exists a limite curve  $\sigma$ , as  $\epsilon \rightarrow 0$ , with the property that

$$L_Y(\sigma) \leq d(p, q) - \alpha/2.$$

Since  $\{f(x_i^{(\epsilon)})\}_{i=0}^N$  are finer and finer on  $\sigma_\epsilon$ , as  $\epsilon \rightarrow 0$ , then  $\sigma \subset f(X)$ . Since  $f$  is a homeomorphism between  $X$  and  $f(X)$ , we have the existence of a curve  $\gamma$  from  $p$  to  $q$  with the property that

$$f \circ \gamma = \sigma.$$

We arrive at a contradiction since

$$\begin{aligned} d(p, q) &\leq L_X(\gamma) \\ &= L_Y(\sigma) \\ &\leq d(p, q) - \alpha/2. \end{aligned}$$

□

### 3.3. Metric spaces that are not path isometrically embeddable.

*Proof of Proposition 1.8.* Fix  $p \in M$ . We prove that the norm  $\|\cdot\|$  at  $p$  is coming from a scalar product by showing that it is the pull back norm of an Euclidean norm via a linear map. Let  $N \subset M$  be a compact neighborhood of  $p$ . Therefore, on  $N$  there exists a Riemannian structure  $g$  such that

$$\sqrt{g_q(v, v)} \geq \|v\|_q, \quad \forall q \in N, \forall v \in T_q M.$$

In other words, on  $N$  we have inequalities of the two distances

$$d_g \geq d_{\|\cdot\|}.$$

If  $f : (M, \|\cdot\|) \rightarrow \mathbb{E}^k$ , is a path isometry, then it is a 1-Lipschitz map. Thus, for all  $x, y \in N$ , we have that

$$d_{\mathbb{E}}(f(x), f(y)) \leq d_{\|\cdot\|}(x, y) \leq d_g(x, y).$$

In other words,  $f : (M, g) \rightarrow \mathbb{E}^k$  is a Lipschitz map between Riemannian spaces. According to Rademacher Theorem, the differential  $df_q$  exists at almost all  $q \in N$ . Namely, there are points  $q \in N$  arbitrary close to the above-fixed point  $p$  with the property that, for all  $v \in T_q M$ , if  $\gamma : [-1, 1] \rightarrow M$  is a curve such that  $\dot{\gamma}(0) = v$ , we have the existence of the limit

$$(df_q)(v) := \lim_{t \rightarrow 0} \frac{(f \circ \gamma)(t) - (f \circ \gamma)(0)}{t}.$$

Moreover, Rademacher Theorem also says that  $(df_q)(v)$  is linear in  $v \in T_q M$ .

Roughly speaking we would like to claim the following. Since  $f$  is a path isometry, it sends infinitesimal balls in  $(M, \|\cdot\|)$  to infinitesimal balls in  $(f(M), d_{\mathbb{E}})$ . However, if



$f$  is differentiable at  $q$  then infinitesimal balls at  $f(q)$  are circles and, being  $df_q$  linear, infinitesimal balls at  $q$  would be ellipses.

More formally, let  $v \in T_q N$  and let  $\gamma : [-\epsilon, \epsilon] \rightarrow M$  a smooth (Finsler) geodesic with  $\dot{\gamma}(0) = v$ , for some  $\epsilon > 0$ . Being geodesic implies that

$$d_{\|\cdot\|}(\gamma(0), \gamma(t)) = t \|v\|, \quad \forall t \in (0, \epsilon).$$

We also have that, as  $t \rightarrow 0$ ,

$$\begin{aligned} d_{\|\cdot\|}(\gamma(0), \gamma(t)) &= L_{\|\cdot\|}(\gamma|_{[0,t]}) \\ &= L_{\mathbb{E}}((f \circ \gamma)|_{[0,t]}) \\ &= d_{\mathbb{E}}((f \circ \gamma)(0), (f \circ \gamma)(t)) + o(t), \end{aligned}$$

since  $f$  is a path isometry and since  $(f \circ \gamma)$  is differentiable at 0. Dividing by  $t$  and considering the limit as  $t \rightarrow 0$ , we get

$$\|v\| = \lim_{t \rightarrow 0} \left\| \frac{(f \circ \gamma)(t) - (f \circ \gamma)(0)}{t} \right\|_{\mathbb{E}} = \|(df_q)(v)\|_{\mathbb{E}}.$$

In other words,  $\|\cdot\|$  at  $q$  is the pull back norm via  $df_q$  of the Euclidean norm  $\|\cdot\|_{\mathbb{E}}$ . Since  $df_q$  is linear, the norm  $\|\cdot\|$  at  $q$  comes from a scalar product. Since we can consider a sequence of points of differentiability  $q$  tending to  $p$ , we also have the same result for the generic  $p$ , by continuity of the Finsler structure.  $\square$

#### 4. LIPSCHITZ EMBEDDINGS FOR FINITE DIMENSIONAL METRIC SPACES

**4.1. Preliminaries.** To prove the Embedding Theorem 1.9, we shall recall the notion of general position. A set  $\{\mathbf{x}_0, \dots, \mathbf{x}_k\}$  of points of  $\mathbb{R}^N$  is said to be *geometrically independent*, or *affinely independent*, if the equations

$$\sum_{j=1}^k a_j \mathbf{x}_j = \mathbf{0} \quad \text{and} \quad \sum_{j=1}^k a_j = 0$$

hold only if each  $a_j = 0$ . In the language of ordinary linear algebra, this is just the definition of linear independence for the set of vectors  $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_k - \mathbf{x}_0$  of the vector space  $\mathbb{R}^N$ . So  $\mathbb{R}^N$  contains no more than  $N + 1$  geometrically independent points.

A set  $A$  of points of  $\mathbb{R}^N$  is said to be in *general position* in  $\mathbb{R}^N$  if every subset of  $A$  is geometrically independent. Observe that, given a finite set  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  of points of  $\mathbb{R}^N$  and given  $\delta > 0$ , there exists a set  $\{\mathbf{y}_1, \dots, \mathbf{y}_n\}$  of points of  $\mathbb{R}^N$  in general position in  $\mathbb{R}^N$ , such that  $|\mathbf{x}_j - \mathbf{y}_j| < \delta$  for all  $j$ .

**Proposition 4.1.** *Suppose  $K$  is a compact subset of  $\mathbb{R}^n$  of Hausdorff dimension  $k$ . If  $n > 2k + 1$ , then there is a full measure subset  $A$  of the unit sphere  $\mathbb{S}^{n-1}$  such that if  $v$  is an element of  $A$ , and*

$$\pi_v : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$$

is the orthogonal projection along  $v$ , then the restriction of  $\pi_v$  to  $K$  is a (Lipschitz) homeomorphism.

*Proof.* The proof is based on the fact that every pair of distinct points in  $K$  determines a line in  $\mathbb{R}^n$ , and hence an element of projective space  $\mathbb{R}P^{n-1} = S^{n-1}/\pm 1$ . The map  $K \times K \setminus \text{Diag}(K \times K) \longrightarrow \mathbb{R}P^{n-1}$  is locally Lipschitz. Thus its image has Hausdorff dimension  $\leq 2k$ . The complement in  $\mathbb{R}P^{n-1}$  gives the set  $A$ .  $\square$

*Remark 4.2.* We can iterate the proposition to conclude that, if  $K$  is a compact  $k$ -dimensional subset of  $\mathbb{R}^n$ , we can find a (full-measure) set of orthogonal projections  $\tilde{\pi} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ , as soon as  $n > m = 2k + 1$ , that are homeomorphisms when restricted to  $K$ .

*Remark 4.3.* Since  $A$  has full measure, it is dense. Thus, given any projection, it is possible to find a ‘good’ projection as close as we want.

The core of the proof in the theorem of Menger and Nöbeling is the construction of embeddings that are close to be injective. One uses the analytic geometry of  $\mathbb{R}^N$  discussed earlier. We present now the relative version for the Lipschitz case.

*Lemma 4.4.* *If  $(X, d)$  is a compact metric space of topological dimension  $m$ , then there exists a Lipschitz map arbitrary close to be injective with range into the Euclidean space of dimension  $N := 2m + 1$ , i.e., for any fixed  $\epsilon > 0$  there exist  $g \in \text{Lip}(X; \mathbb{R}^N)$  such that*

$$g(x_1) = g(x_2) \implies d(x_1, x_2) < \epsilon.$$

*Proof.* By the definition of topological dimension, we have that we can cover  $X$  by finitely many open sets  $\{U_1, \dots, U_n\}$  such that

- (1)  $\text{diam } U_j < \epsilon$  in  $X$ ,
- (2)  $\{U_1, \dots, U_n\}$  has order  $\leq m + 1$ .

The second requirement means that no point of  $X$  lies in more than  $m + 1$  elements of the cover.

Let  $\phi_j$  be a Lipschitz partition of unity dominated by  $\{U_j\}$ , cf. [LV77]. For each  $j$ , choose a point  $\mathbf{z}_j \in \mathbb{R}^N$  such that the set  $\{\mathbf{z}_1, \dots, \mathbf{z}_n\}$  is in general position in  $\mathbb{R}^N$ . Finally, define  $g : X \longrightarrow \mathbb{R}^N$  by the equation

$$g(x) = \sum_{j=1}^n \phi_j(x) \mathbf{z}_j.$$

We assert that  $g$  is the desired function.

At every point  $x$ , locally  $g(x)$  is a sum of finitely many Lipschitz maps, thus is Lipschitz.

We shall prove that if  $x_1, x_2 \in X$  and  $g(x_1) = g(x_2)$ , then  $x_1$  and  $x_2$  belong to one of the open sets  $U_j$ , so that necessarily  $d(x_1, x_2) < \epsilon$  (since  $\text{diam } U_j < \epsilon$ ).

So suppose  $g(x_1) = g(x_2)$ . Then

$$\sum_{j=1}^n [\phi_j(x_1) - \phi_j(x_2)] \mathbf{z}_j = 0.$$

Because the covering  $\{U_j\}_{j=1}^n$  has order at most  $m+1$ , at most  $m+1$  of the numbers  $\{\phi_j(x_1)\}_{j=1}^n$  are nonzero, and at most  $m+1$  of the numbers  $\{\phi_j(x_2)\}_{j=1}^n$  are nonzero. Thus, the sum  $\sum [\phi_j(x_1) - \phi_j(x_2)] \mathbf{z}_j = 0$  has at most  $2m+2$  nonzero summands. Note that the sum of the coefficients vanishes because

$$\sum [\phi_j(x_1) - \phi_j(x_2)] = 1 - 1 = 0.$$

The points  $\mathbf{z}_j$ , are in general position in  $\mathbb{R}^N$ , so that any subset of them having  $N+1$  or fewer elements is geometrically independent. And by hypothesis  $N+1 = 2m+2$ . (Aha!) Therefore, we conclude that

$$\phi_j(x_1) - \phi_j(x_2) = 0$$

for all  $j$ . Now  $\phi_j(x_1) > 0$  for some  $j$ , so that  $x_1 \in U_j$ . Since  $\phi_j(x_1) - \phi_j(x_2) = 0$ , we have that  $x_2 \in U_j$  as well, as asserted.  $\square$

**4.2. The proof of the Embedding Theorem 1.9.** Let  $N = 2m+1$ . Consider the space  $\text{Lip}(X; \mathbb{R}^N)$ , i.e., the space of all the Lipschitz maps from  $X$  to  $\mathbb{R}^N$  (It's non empty, since the constants are there). It is complete in the following metric

$$\|f\|_{\text{Lip}} := \|f\|_{\infty} + \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\}.$$

Let  $d$  be the metric of the space  $X$ ; because  $X$  is compact,  $d$  is bounded. Given a map  $f : X \rightarrow \mathbb{R}^N$ , let us define

$$\Delta(f) := \sup \{ \text{diam } f^{-1}(\mathbf{z}) : \mathbf{z} \in \mathbb{R}^N \},$$

i.e., the fibers of  $f$  have diameter smaller than  $\Delta(f)$ . So the number  $\Delta(f)$  measures how far  $f$  is far from being injective; if  $\Delta(f) = 0$ , then in fact  $f$  is injective.

Now, given  $\epsilon > 0$ , define  $\mathcal{U}_{\epsilon}$  to be the set of all those Lipschitz maps  $f : X \rightarrow \mathbb{R}^N$  for which  $\Delta(f) < \epsilon$ . In Lemma 4.5 and in Lemma 4.6 we shall show that  $\mathcal{U}_{\epsilon}$  is both open and dense in  $\text{Lip}(X; \mathbb{R}^N)$ , respectively. So it follows from Baire Category Theorem that the intersection

$$\bigcap_{n \in \mathbb{N}} \mathcal{U}_{\frac{1}{n}}$$

is dense in  $\text{Lip}(X; \mathbb{R}^N)$  and is in particular non-empty. If  $f$  is an element of this intersection, then  $\Delta(f) < 1/n$  for every  $n$ . Therefore,  $\Delta(f) = 0$  and  $f$  is injective. Because  $X$  is compact,  $f$  is an embedding. Thus, modulo the Lemma 4.5 and Lemma 4.6, the theorem is proved.  $\square$

*Lemma 4.5.*  $\mathcal{U}_\epsilon$  is open in  $\text{Lip}(X; \mathbb{R}^N)$ .

Given an element  $f \in \mathcal{U}_\epsilon$ , we wish to find some ball about  $f$  that is contained in  $\mathcal{U}_\epsilon$ . First choose a number  $b$  such that  $\Delta(f) < b < \epsilon$ . Let  $A$  be the following subset

$$A = \{(x, y) \in X \times X \mid d(x, y) \geq b\}.$$

Now  $A$  is closed in  $X \times X$  and therefore compact.

Note that if  $f(x) = f(y)$ , then  $d(x, y)$  must be less than  $b$ . It follows that the function  $|f(x) - f(y)|$  is positive on  $A$ . Since  $A$  is compact, the function has a positive minimum on  $A$ . Let

$$\delta := \frac{1}{2} \min \{|f(x) - f(y)| : x, y \in A\}.$$

We assert that this value of  $\delta$  will suffice.

Suppose that  $g$  is a map such that  $\|f - g\|_{\text{Lip}} < \delta$ . So in particular  $\|f - g\|_\infty < \delta$ . If  $(x, y) \in A$ , then  $|f(x) - f(y)| > 2\delta$  by definition of  $\delta$ . Since  $g(x)$  and  $g(y)$  are within  $\delta$  of  $f(x)$  and  $f(y)$ , respectively, we must have that  $|g(x) - g(y)| > 0$ . Hence the function  $|g(x) - g(y)|$  is positive on  $A$ . As a result, if  $x$  and  $y$  are two points such that  $g(x) = g(y)$ , then necessarily  $d(x, y) < b$ . We conclude that  $\Delta(g) \leq b < \epsilon$ , as desired.  $\square$

*Lemma 4.6.*  $\mathcal{U}_\epsilon$  is dense in  $\text{Lip}(X; \mathbb{R}^N)$ .

This is the more substantial part of the proof. We shall use the preliminaries presented in the previous subsection. Let  $f \in \text{Lip}(X; \mathbb{R}^N)$ . Given  $\delta > 0$ , we wish to find a function  $F \in \text{Lip}(X; \mathbb{R}^N)$  such that  $F \in \mathcal{U}_\epsilon$  and  $\|f - F\|_{\text{Lip}} < \delta$ .

Since the topological dimension of  $X$  is smaller than  $m$ , we can apply Lemma 4.4. Take  $g \in \text{Lip}(X; \mathbb{R}^N)$  such that if  $g(x_1) = g(x_2)$  then  $d(x_1, x_2) < \epsilon/2$ .

Consider  $\Phi := (f, g) : X \longrightarrow \mathbb{R}^{2N}$ . Clearly,  $\Phi$  is Lipschitz. Thus,  $\Phi(X)$  has Hausdorff dimension no more than  $m$ .

Since  $2N > N = 2m + 1$ , we can use Proposition 4.1 (and the remarks afterwards) to projections from  $\mathbb{R}^{2N}$  to  $\mathbb{R}^N$  and the compact set  $K = \Phi(X)$ . Namely, there are orthogonal projections that are injective on  $K$  and are arbitrarily close to the projection in the first  $N$ -dimensional component. Explicitly, for any  $\beta > 0$ , there exists an orthogonal projection  $\tilde{\pi} : \mathbb{R}^{2N} \longrightarrow \mathbb{R}^N$  such that the restriction of  $\tilde{\pi}$  to  $K$  is a (Lipschitz) homeomorphism and, if  $\pi : \mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$  is given by  $\pi(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ , then

$$\|\tilde{\pi} - \pi\| < \beta.$$

We are using here the operator norm. We will say later how much small  $\beta$  has to be in terms of the data  $(f, g, \delta)$ .

Set  $F := \tilde{\pi} \circ \Phi$ . We shall prove first that  $F \in \mathcal{U}_\epsilon$  and then  $\|f - F\|_{\text{Lip}} < \delta$ .

Suppose  $x_1, x_2$  are in the same fiber of  $F$ , i.e.,  $F(x_1) = F(x_2)$ . So from the definition of  $F$ ,

$$(\tilde{\pi} \circ \Phi)(x_1) = (\tilde{\pi} \circ \Phi)(x_2)$$

Since  $\tilde{\pi}$  is a homeomorphism on  $K = \Phi(X)$ , we have that

$$\Phi(x_1) = \Phi(x_2).$$

From the definition of  $\Phi$ , we have that

$$(f(x_1), g(x_1)) = (f(x_2), g(x_2)).$$

In particular,  $g(x_1) = g(x_2)$ . So, by the property of  $g$ , we have that  $d(x_1, x_2) < \epsilon/2$ . Therefore,  $F \in \mathcal{U}_\epsilon$ .

Let us prove now that  $F$  is  $\delta$ -close to  $f$ . Let us write explicitly the difference

$$\begin{aligned} F(x) - f(x) &= (\tilde{\pi} \circ \Phi)(x) - f(x) \\ &= \tilde{\pi}(f(x), g(x)) - \pi(f(x), g(x)) \\ &= (\tilde{\pi} - \pi)(f(x), g(x)). \end{aligned}$$

Bound the sup norm by

$$\begin{aligned} |F(x) - f(x)| &\leq \|\tilde{\pi} - \pi\| |(f(x), g(x))| \\ &\leq \|\tilde{\pi} - \pi\| \sqrt{\|f\|_\infty^2 + \|g\|_\infty^2} \\ &\leq \beta \sqrt{\|f\|_{Lip}^2 + \|g\|_{Lip}^2}. \end{aligned}$$

For the Lipschitz part of the norm, remember that the projections are linear. Therefore

$$\begin{aligned} \frac{|F(x) - f(x) - (F(y) - f(y))|}{|d(x, y)|} &\leq \frac{|(\tilde{\pi} - \pi)(f(x), g(x)) - (\tilde{\pi} - \pi)(f(y), g(y))|}{d(x, y)} \\ &\leq \frac{|(\tilde{\pi} - \pi)(f(x) - f(y), g(x) - g(y))|}{d(x, y)} \\ &\leq \|\tilde{\pi} - \pi\| \frac{|(f(x) - f(y), g(x) - g(y))|}{d(x, y)} \\ &\leq \|\tilde{\pi} - \pi\| \sqrt{\|f\|_{Lip}^2 + \|g\|_{Lip}^2} \\ &\leq \beta \sqrt{\|f\|_{Lip}^2 + \|g\|_{Lip}^2}. \end{aligned}$$

So choose  $\beta$  such that  $\beta \sqrt{\|f\|_{Lip}^2 + \|g\|_{Lip}^2} < \delta/2$ . □

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